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1997 J. Phys. A: Math. Gen. 30 7997

(http://iopscience.iop.org/0305-4470/30/23/004)

J. Phys. A: Math. Gen. **30** (1997) 7997–8015. Printed in the UK PII: S0305-4470(97)85814-3

# Universality classes for extreme-value statistics

Jean-Philippe Bouchaud†§ and Marc Mézard‡‖

† Service de Physique de l'État Condensé, Centre d'études de Saclay, Orme des Merisiers, 91191 Gif-sur-Yvette Cedex, France

‡ Laboratoire de Physique Théorique de l'Ecole Normale Supérieure¶, 24 rue Lhomond, 75231 Paris Cedex 05, France

Received 9 July 1997

**Abstract.** The equilibrium low-temperature physics of disordered systems is governed by the statistics of extremely low-energy states. It is thus relevant to discuss the possible universality classes for extreme-value statistics. We compare the usual probabilistic classification to the results of the replica approach. We show in detail for several problems (including the random energy model and the decaying Burgers turbulence) that one class of independent variables corresponds exactly to the so-called *one step replica symmetry breaking* solution in the replica language. We argue that this universality class holds if the correlations are sufficiently weak, and propose a conjecture on the level of correlations which leads to different universality classes.

## 1. Introduction

The replica method is one of the very few general analytical methods available to investigate disordered systems [1]. Although the physical meaning of Parisi's 'replica symmetry breaking' (RSB) scheme needed to obtain the correct low-temperature solution of various random models has already been discussed on several occasions [1], its precise relation with the so-called extreme-value statistics [2, 3] (and therefore its scope and limitations) was not previously clearly established. That such a relation should exist is however intuitively obvious: at low temperatures, a disordered system will preferentially occupy its low-energy states, which are random variables because of the disordered nature of the problem. The statistics of the free-energy (or of other observables, such as energy *barriers* [4, 5]) will thus reflect the statistics of these low-energy (extreme) states. It is well known in probability theory that extreme-value statistics can be classified into different universality classes [2, 3]. Conversely, the RSB scheme has shown the existence of at least two broad classes of systems, those with a first-order, 'one-step' RSB and those with continuous RSB.

It is easy to identify the 'one-step RSB' class with one particular universality class of extreme-value statistics, i.e. the Gumbel class, which concerns the minimum of continuous variables which are unbounded but have a distribution decaying faster than any power at $-\infty$. The simplest representative of this class is the random energy model (REM) [6], where the energy states are independent (but not necessarily Gaussian distributed). An interesting point is that the REM can be given a spatial structure, for which the replica theory still provides the *exact* solution. This spatial REM is in turn connected, in one dimension, to

§ E-mail address: bouchaud@amoco.saclay.cea.fr

‖ E-mail address: mezard@physique.ens.fr

¶ Unité propre du CNRS, associée à l'Ecole Normale Supérieure et à l'Université de Paris Sud.

0305-4470/97/237997+19$19.50  

the problem of decaying Burgers turbulence [8] in which an infinitely compressible fluid evolves from random initial conditions. Exact results for the velocity correlations at large times in Burgers turbulence were obtained long ago by Kida [9]. We shall show that these results coincide with those of the replica method, the underlying reason being that the late stage of turbulence decay is governed by the extreme values for the integral of the initial velocity field.

It is less easy to identify the other universality classes of extreme-value statistics. There should be at least two types of generalizations. One type still concerns *independent* random variables but with either power-law decay of the distribution (in which case there is *a priori* no replica formalism), or bounded random variables (the Weibull distribution of extremes), which does not seem to correspond to any known RSB scheme. The other type concerns *correlated* variables, for which the only results known to us are actually derived in the framework of replicas: those are cases of full RSB, which describe random variables with a certain (hierarchical) type of correlations.

These universality classes are the counterpart for extremes of random variables to the usual universality classes studied in the framework of sums of random variables. Taking the well known example of random walks or polymers, the usual random walk, or ideal polymer, is described asymptotically by the Gaussian central-limit theorem, while the addition of independent variables with infinite variance leads to new universality classes (Lévy sums) [10]. The introduction of long-range correlations such as self avoidance also leads to totally new universality classes [10]. We wish here to take a first step in an analogous categorization for extreme values, which appear naturally in disordered systems at low temperatures. We thus often rely on existing results and put them in this perspective. While the aim of this paper is mainly methodological, it also contains the following new results: a version of the REM with a first-order phase transition (section 3.2), or with no transition (section 6), a reformulation of Kida's results on the statistics of shocks in declining Burgers turbulence using replicas (section 5.3), and a conjecture on the amount of correlations we can add to the REM without changing the universality class of the low-temperature phase (section 6).

## 2. Extreme-value statistics

### *2.1. Scaling regime*

We start by recalling standard results of extreme-values statistics, in order to set the stage for the following discussions. Consider $M$ independent, identically distributed random variables $E_i, i = 1, \ldots, M$ ('energies'), such that the probability distribution decays for $E_i \to -\infty$ faster than any power law:

$$P(E) \sim \frac{A}{|E|^{\alpha}} \exp[-B|E|^{\delta}] \qquad B, \delta > 0 \qquad E \to -\infty. \tag{1}$$

We are interested in the statistics of the lowest energy state $E^* = \min\{E_1, \ldots E_M\}$ for large $M$. Defining $\mathcal{P}_<(E)$ as the repartition function of $E$

$$\mathcal{P}_<(E) = \int_{-\infty}^{E} \mathrm{d}E'\, P(E') \tag{2}$$

we can express the distribution $P_M$ of $E^*$ as

$$P_M(E^*) = M P(E^*)[1 - \mathcal{P}_<(E^*)]^{M-1} = -\frac{\mathrm{d}}{\mathrm{d}E^*}[\mathcal{P}_>(E^*)]^M \tag{3}$$

For large $M$, the minimum $E^*$ will be negative and large, so that

$$[1 - \mathcal{P}_<(E^*)]^M \simeq \exp[-M\mathcal{P}_<(E^*)]. \tag{4}$$

The repartition function of $E^*$ thus becomes very small when $E^*$ is smaller than the characteristic value of the energy $E_c(M)$ defined by $M\mathcal{P}_<(E_c) = 1$. To logarithmic accuracy, this gives in the case of the distribution (1)

$$E_c(M) \simeq -\left(\frac{\log M}{B}\right)^{1/\delta}. \tag{5}$$

The value of the extreme $E^*$ fluctuates around $E_c$. By expanding (4) in $(E^* - E_c)/E_c$, one sees that the size of fluctuations is

$$\Delta(M) = \frac{1}{B\delta|E_c|(M)}. \tag{6}$$

More precisely, the rescaled minimum energy variable $u \equiv (E^* - E_c(M))/\Delta(M)$ obeys, for large $M$, a universal 'Gumbel' distribution [2, 3]

$$P^*(u) = \exp(u - \exp u). \tag{7}$$

The maximum of $P^*(u)$ occurs at $u = 0$, meaning that $E_c(M)$ is actually the most probable value for the extreme energy. Finally, as in any 'central'-limit theorem, this behaviour is only valid in the region where the deviation $\epsilon$ from $E_c$ is of the order of $\Delta(M)$, which goes to zero with $M$ if $\delta > 1$ and diverges otherwise. The *relative* fluctuations $\Delta(M)/E_c(M)$, however, are always of order $1/\log M$. A very important property, which we shall emphasize later on, is that $P^*(u)$ vanishes exponentially for $u \to -\infty$. In the scaling region, the probability for a given energy $E_i$ to be $E_i = E_c(M) + \Delta(M)u_i$ behaves as $(c/M)\exp(u_i)$. So the low-lying energies are independent random variables which are exponentially distributed.

### *2.2. The large M limit and the random energy model*

Let us now consider the following partition function:

$$\mathcal{Z} = \sum_{i=1}^{M} z_i \qquad z_i = \exp\left[-\frac{E_i}{T}\right] \tag{8}$$

where the $E_i$ are distributed as in (1). This is a slight generalization of Derrida's original REM, initially introduced with a purely Gaussian distribution ($\delta = 2$). Obviously, the independent variables $z_i$, are large when $E_i$ is large and negative. In the scaling region defined above, the exponential distribution of the rescaled energy translates into a power-law decay of $P(z)$ for large $z$:

$$P(z) \propto z^{-1-\mu} \qquad z \to \infty \tag{9}$$

where $\mu = T/\Delta(M)$. The partition sum $\mathcal{Z}$ behaves very differently in the region $\mu < 1$, where the average value of $z$ diverges and thus only a small number of terms (those of order $M^{1/\mu}$) contribute to $\mathcal{Z}$, and in the region $\mu > 1$, where all the $M$ terms give a (small) contribution to $\mathcal{Z}$. This means that for

$$T_c = \Delta(M) \tag{10}$$

for which $\mu = 1$, the probability measure concentrates onto a finite number of states, corresponding to the glass transition in these models. In the random-energy model, $M$ is the number of states $M = 2^N$. By choosing $B = N^{1-\delta}$, it is possible to ensure both an

extensive ground-state energy ($E_c \propto N$) and a finite $T_c$ in the large $N$ limit. Let us now study the statistics of the weights $p_i \equiv z_i/\mathcal{Z}$ in the glassy region $T < T_c$. Since

$$w_i = \frac{z_i}{z_i + \mathcal{Z}'} \tag{11}$$

where $\mathcal{Z}' = \sum_{k(\neq i)} z_k$ is independent of $z_i$ (and of order $M^{1/\mu}$), one readily finds that†

$$P(w) = \frac{\mathcal{Z}'}{(1-w)^2} P\left(z = \frac{\mathcal{Z}' w}{1-w}\right). \tag{12}$$

For $w_i$ to be nonzero in the large $M$ limit, $z_i$ has to be large. In that region we can use the asymptotic form (9) for $P(z)$, giving

$$P(w) = \frac{C}{M}(1-w)^{\mu-1} w^{-1-\mu} \qquad w \gg M^{-1/\mu} \tag{13}$$

where $C$ is a constant fixed by the condition $M \int_0^1 \mathrm{d}w\, w P(w) \equiv 1$. From this probability distribution of each weight, we can deduce the moments $Y_k \equiv \overline{\sum_i w_i^k}$, which characterize to what extent the measure concentrates onto a few states: if all weights are of the same order of magnitude, then $Y_k \sim M^{1-k} \to 0$ for $k > 1$; while if only a finite number of weights contribute, the moments $Y_k$ remain finite when $M \to \infty$. In this case, one finds, for $\mu < 1$,

$$Y_k = M \int_0^1 \mathrm{d}w\, w^k P(w) = \frac{\Gamma[k-\mu]}{\Gamma[k]\Gamma[1-\mu]} \qquad (k > \mu) \tag{14}$$

(see also [11]). Since $\mu = T/T_c$, one finds that $Y_2$ goes linearly to zero for $T \to T_c$, and that $Y_k = 1 - (\Gamma'[k] - \Gamma'[1])/\Gamma[k])T/T_c$ for $T \to 0$. Finally, the average energy per degree of freedom of the system is constant throughout the low-temperature phase ($T < T_c$) and given by $\overline{E}/N = E_c/N + \mathrm{O}(1/N) \sim -(\log 2)^{1/\delta}$.

## 3. The replica approach

### *3.1. The REM*

We shall now show how all these results can be recovered using the replica method. We suppose that $\delta > 1$ (the case $\delta < 1$ will be discussed in section 3.2) and introduce the characteristic function $g(\lambda)$ through

$$\int_{-\infty}^{\infty} \mathrm{d}E\, P(E) \exp[-\lambda E] \equiv \exp[g(\lambda)]. \tag{15}$$

Since $B = N^{1-\delta}$, this integral can be computed at large $N$ with a saddle-point method, which gives

$$g(\lambda) = (\delta - 1) N \left(\frac{\lambda}{\delta}\right)^{\delta/\delta - 1}. \tag{16}$$

In the replica method we need to compute the moments of the $\mathcal{Z}$ distribution:

$$\overline{\mathcal{Z}^n} = \overline{\sum_{i_1, i_2, \ldots i_n} z_{i_1} z_{i_2} \ldots z_{i_n}} \equiv \overline{\sum_{i_1, i_2, \ldots i_n} \exp\left[ -\frac{1}{T} \sum_i E_i \sum_{a=1}^{n} \delta_{i, i_a} \right]}. \tag{17}$$

Averaging over the $E_i$ gives

$$\overline{\mathcal{Z}^n} = \sum_{i_1, i_2, \ldots i_n} \exp\left[ \sum_i g\left( \frac{1}{T} \sum_{a=1}^{n} \delta_{i, i_a} \right) \right]. \tag{18}$$

† We denote as $P(.)$ the probability density of the variable appearing in the parentheses.

The point now is to understand which configurations of $\{i_1, i_2, \dots i_n\}$ will dominate the above sum when $N \to \infty$ (and $n \to 0$). The simplest Ansatz, corresponding to the largest phase-space volume, assumes that all $i_a$ are different, leading to

$$\overline{\mathcal{Z}^n} = M(M-1)\dots(M-n+1)\exp\left[ng\left(\frac{1}{T}\right)\right] \simeq \exp\left[n\left(\log M + g\left(\frac{1}{T}\right)\right)\right]. \tag{19}$$

Taking $n \to 0$, one thus finds that the free energy per degree of freedom $f = -\frac{T}{N}\overline{\log \mathcal{Z}}$ takes the value

$$f = f_0 \equiv -T\log 2 - (\delta - 1)\delta^{-\frac{\delta}{\delta-1}} T^{-\frac{1}{\delta-1}}. \tag{20}$$

The entropy $s_0 = -\mathrm{d}f_0/\mathrm{d}T$ is therefore equal to

$$s_0 = \log(2) - (\delta T)^{-\frac{\delta}{\delta-1}} \tag{21}$$

and becomes negative below a critical temperature

$$T_c = \frac{1}{\delta}\log(2)^{\frac{1-\delta}{\delta}}. \tag{22}$$

So this solution, called 'replica symmetric' (since all replicas $i_a$ play a symmetric role), has to be modified in the low-temperature phase. The correct configurations which dominate the sum (18) at $T < T_c$ are called 'one-step RSB' and are such that the $n$ replica indices $\{i_1, i_2, \dots i_n\}$ are grouped into $n/m$ groups of $m$ equal indices, which can be written, after proper relabelling:

$$\begin{aligned} &i_1 = i_2 = \dots = i_m = k_1 \\ &i_{m+1} = i_{m+2} = \dots = i_{2m} = k_2 \\ &\dots \\ &i_{n-m+1} = \dots = i_n = k_{n/m} \end{aligned} \tag{23}$$

and now the indices $k_1, \dots, k_{n/m}$ are all different one from the other. These configurations contribute to $\overline{\mathcal{Z}^n}$ as

$$\overline{\mathcal{Z}^n} = M(M-1)\dots(M-n/m+1)\exp\left[\frac{n}{m}g\left(\frac{m}{T}\right)\right]\frac{n!}{m!^{n/m}} \tag{24}$$

from which one immediately deduces

$$f(T) = f_0(T/m) \tag{25}$$

where $f_0$ is defined in equation (20). The extremum of this free energy with respect to $m$ is obtained when

$$\frac{\partial f}{\partial m} = 0 = s_0(T/m) \tag{26}$$

which gives

$$m = \frac{T}{T_c} = \mu. \tag{27}$$

Note that this relation is independent of $\delta$. Therefore this one-step RSB solution predicts that the system freezes at the critical temperature $T_c$ which is the temperature where the entropy $s_0$ vanishes. The energy density is constant throughout the low-temperature phase, and equals

$$e = f_0(T_c) = -(\log 2)^{1/\delta} \tag{28}$$

in agreement with the direct computation. Since the free-energy is constant, the entropy of the whole low-temperature phase is zero [6].

It turns out that the finer details, like the distribution of the weights of the configurations which dominate the low-temperature measure, can be computed by this replica approach [12]. By definition, the moments $Y_k$ are equal to

$$Y_k = \overline{\sum_i \frac{z_i^k}{\mathcal{Z}^k}} = \lim_{n\to 0} \overline{\sum_i z_i^k \mathcal{Z}^{n-k}} \tag{29}$$

$$= \lim_{n\to 0} \frac{1}{n(n-1)\dots(n-k+1)} \sum_{a_1,\dots,a_k}{}' \sum_{i_1,\dots,i_n} \overline{z_{i_1}\dots z_{i_n}} \prod_{j=1}^{k} \delta_{i_{a_1},i_{a_j}} \tag{30}$$

where the sum primed over the $a$'s runs from 1 to $n$, with all $a$'s different. Owing to the structure of the RSB, this means that one simply has to pick the $k \leqslant m$ replica indices $a_1, \dots, a_k$ in the same 'group', for which there are $(m-1)\dots(m-k+1)$ possibilities once $a_1$ has been chosen. Hence,

$$Y_k = \lim_{n\to 0} \frac{n(m-1)\dots(m-k+1)}{n(n-1)\dots(n-k+1)} \overline{\mathcal{Z}^n} = \frac{\Gamma[k-\mu]}{\Gamma[k]\Gamma[1-\mu]} \tag{31}$$

in agreement with the direct computation (14).

### *3.2. The REM with $\delta < 1$: a first-order transition*

The above method fails when $\delta \leqslant 1$, which actually corresponds to a different universality class from the point of view of critical phenomena, while the nature of the low-temperature phase leaves it in the same class as the systems with $\delta > 1$, in agreement with the extreme-value classification which does not distinguish between $\delta > 1$ or $\delta < 1$. In order to study the transition, we use Derrida's original 'microcanonical' method. Using the normalization $B = N^{1-\delta}$, the partition function is equal to

$$\mathcal{Z} = \int_{-e_c}^{0} \mathrm{d}e \, \exp[N\varphi(e)] \qquad \varphi(e) = \log 2 - |e|^\delta + \frac{|e|}{T} \tag{32}$$

where $e = E/N$, and $e_c$ is the energy density beyond which there are no states (for $N \to \infty$), given by $2^N \exp(-Ne_c^\delta) = 1$. As shown in figure 1, the integral is dominated either by $e = 0$ or by $e = -e_c$, depending on the temperature. When $T > T_c = (\log 2)^{1-\delta/\delta}$, the free energy is equal to $-NT\log 2$, while for $T < T_c$, the free energy is equal to a constant $-Ne_c = -N(\log 2)^{1/\delta}$. The transition at $T_c$ is now a first-order transition from the thermodynamic point of view, with a jump in the entropy. This is in contrast to the usual case $\delta > 1$ where the transition is thermodynamically of second order†.

In the low-temperature phase, only the neighbourhood of $E_c = -N|e_c|$ is of importance, and we get back to the universal Gumbel distribution since the density of states is still locally exponential:

$$P(E = E_c + \epsilon) \propto \exp \frac{\mu\epsilon}{T} \qquad \mu \equiv \frac{T}{T_c}\delta. \tag{33}$$

The value of $\mu$ again determines the statistics of the weights, as above. Note, however, that for $\delta < 1$, the value of the parameter $\mu$ (corresponding to the RSB parameter) is smaller than 1 at the transition $T = T_c^-$. Hence $Y_2$ is discontinuous at $T = T_c$, in contrast to the case $\delta > 1$.

† Although there is a jump in the Edwards–Anderson order parameter [6].

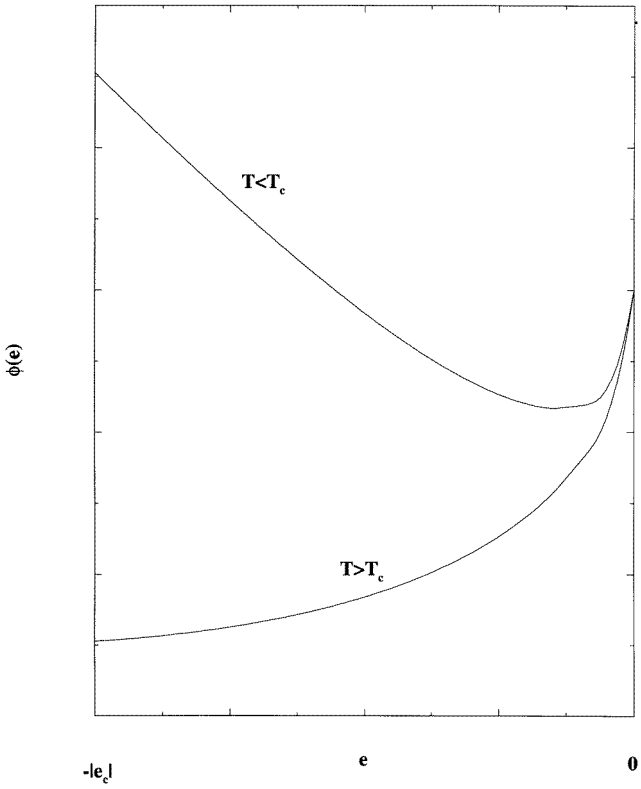


**Figure 1.** $\varphi(e)$, as defined in equation (32), as a function of $e$ for different temperatures. The saddle point is thus at $e = 0$ for $T > T_c$ and at $e = -|e_c|$ for $T < T_c$. There are no states (in the limit $N \to \infty$ beyond $-|e_c|$.

### *3.3. Physical interpretation of the replica solution*

The reason why replica symmetry must be broken in order to get sensible results in this problem is rather clear. Since the distribution of the Boltzmann weights $z_i$ is a power law with an exponent $\mu < 1$ in the low-temperature phase, all integer moments of $\mathcal{Z}$ (and thus of $z$) are formally divergent, and are thus dominated by a cut-off for large $z$ which has nothing to do with the value of the 'typical' $z$'s, and hence of the free-energy. Calculation based on a simple analytic continuation of the results obtained for $n > 1$ are thus bound to fail. The replica method with one step RSB manages to compute $\langle z^m \rangle$ with $m = \mu$, which precisely picks up the contribution of the typical region of $z$. (Smaller values of $m$ would be mostly sensitive to very small $z$, while larger $m$'s probe atypically large values of $z$.) The algebra corresponding to one-step RSB exactly reproduces the extreme-value statistics in the case of fast decaying distributions. In this respect, RSB does not mean more than a 'localization' of weights onto a small subset of all configurations, in the sense that major contribution to the partition function comes from a finite number of configurations (i.e. all $Y_k > 0$) [12]. Actually, the quantities $Y_k$ were also introduced in the context of electron localization in disordered potentials, and called 'participation ratios' [13].

## 4. A d-dimensional random energy model

In this section we want to study a generalized version of the REM, where the energy levels are embedded in a Euclidean space. Besides its intrinsic interest as a model for a particle in a disordered environment, this problem also turns out to be directly relevant to the study of declining Burgers turbulence, as we shall discuss in detail in the next section.

The model is defined as follows. To each point $x$ of a (discretized) $d$-dimensional space, one assigns a potential energy $E(\boldsymbol{x})$ which is a random number picked up independently on each point, from a distribution $P(E)$ the tail of which is given by (1). The total energy on this point is the sum of a deterministic part, which we take for instance equal to $\kappa x^2/2$, and this random contribution $E(\boldsymbol{x})$. This defines a certain energy landscape, to which we associate a partition function $\mathcal{Z}$ as

$$\mathcal{Z} = \int \mathrm{d}^d x \, \exp\left(-\frac{V(\boldsymbol{x})}{T}\right) \qquad V(\boldsymbol{x}) \equiv \frac{\kappa \boldsymbol{x}^2}{2} + E(\boldsymbol{x}). \tag{34}$$

Here we adopt a continuum notation but an ultraviolet cut-off (lattice spacing) is implicitely assumed when necessary. The role of the deterministic part proportional to $\kappa$ is two-fold. First of all it allows us to define a topology in the space of the points $\boldsymbol{x}$ (The limit $\kappa = 0$ coincides with the REM, the fact that the points sit in a $d$-dimensional space is irrelevant). For this purpose the deterministic part could be rather arbitrary, and indeed we can solve the problem with a more general deterministic energy. As we shall need a quadratic term later on, and in order to keep the presentation simple, we restrict ourselves to this particular case. Second, the presence of this confining term allows us to deal with this model without the need to introduce a finite box.

This model with $d = 1$ was in fact introduced and studied long ago as a toy model of an interface in a random medium [14]: one possible interpretation is that $x$ is the coordinate of a particular point on the interface, which feels a random pinning potential $E(x)$, while the quadratic potential is a mean-field description of the elasticity due to the rest of the interface. Another interpretation (in the context of Bloch walls) is to neglect the deformation of the interface, which is only described by its centre of mass coordinate $x$. The quadratic potential is then induced by the demagnetizing fields due to the surrounding Bloch walls.

We want to compute the low-temperature properties of this system in the limit when $\kappa \to 0$. For instance one would like to know the typical displacement of the ground state, measured through $\overline{\langle x^2 \rangle}$, or the average ground-state energy, etc. In the special case where the energy is Gaussian distributed, this problem has already been studied by scaling arguments [14], or with a Gaussian replica variational method [15]. We shall provide hereafter the exact solution, first using a direct extreme-value statistics approach and then with the replica method.

### *4.1. Extreme value approach*

For simplicity, we restrict ourselves to the case $d = 1$; the extension to higher dimensions is however immediate. For temperatures going to zero, we want to find the minimum of all the energies $\frac{\kappa}{2}x^2 + E(x)$ when $x$ scans a one-dimensional lattice. The joint probability that this minimum is achieved on a point $x^*$ and takes a value $V(x) = \frac{\kappa}{2}x^{*2} + E$ is given by

$$P(x^*, E) = P(E) \prod_{x' \neq x^*} \left(1 - P_<\left(E + \frac{\kappa}{2}x^{*2} - \frac{\kappa}{2}x'^2\right)\right). \tag{35}$$

For $\kappa \to 0$ we can safely take a continuum limit and we obtain

$$P(x^*, E) = \frac{P(E)}{1 - P_<(E)} \exp\left( \int \mathrm{d}x' \, \log\left[ 1 - P_<\left( E + \frac{\kappa}{2} x^{*2} - \frac{\kappa}{2} x'^2 \right) \right] \right). \tag{36}$$

Integrating over $E$ we get the probability that the minimum is achieved on point $x^*$. For small $\kappa$, the minimum $E$ is expected to be negative and large, and hence only the region where $P_<$ is small will be of importance. Rescaling $x^*$ as $x^* = \hat{x}^*/\sqrt{\kappa}$, we obtain

$$P(\hat{x}^*) \simeq \int \mathrm{d}E \; P(E) \exp\left( - \int \frac{\mathrm{d}z}{\sqrt{\kappa}} P_<\left( E + \frac{\hat{x}^{*2} - z^2}{2} \right) \right). \tag{37}$$

For small $\kappa$ it is thus clear that the relevant energy region is the one around the value $E_\mathrm{c}$ such that $P_<(E_\mathrm{c}) = \sqrt{\kappa}$, or

$$E_\mathrm{c} = -\left( \frac{\log(1/\sqrt{\kappa})}{B} \right)^{1/\delta}. \tag{38}$$

(Note that the role played here by the number $M$ of energy levels in the first section is the length scale $1/\sqrt{\kappa}$, which is natural.) Expanding the energy around $E_\mathrm{c}$ as $E = E_\mathrm{c} - \hat{x}^{*2}/2 + \epsilon$, we obtain

$$P_<\left( E + \frac{\hat{x}^{*2} - z^2}{2} \right) \sim \sqrt{\kappa} \exp\left( \delta B |E_\mathrm{c}|^{\delta - 1} \left( \epsilon - \frac{z^2}{2} \right) \right). \tag{39}$$

The integral over $z$ in (37) is a thus a Gaussian integral. We finally obtain, after a simple integration over $\epsilon$

$$P(\hat{x}^*) \propto \exp\left( -\delta B |E_\mathrm{c}|^{\delta - 1} \frac{\hat{x}^{*2}}{2} \right). \tag{40}$$

Therefore we have shown that the typical distance to the origin of the point $x^*$ corresponding to a minimum energy is

$$\xi = (\kappa \delta B |E_\mathrm{c}|^{\delta - 1})^{-1/2} = \frac{1}{\sqrt{\kappa \delta}} \left( \log\left( \frac{1}{\sqrt{\kappa}} \right) \right)^{1 - \frac{\delta}{2\delta}} B^{-\frac{1}{2\delta}} \tag{41}$$

and more precisely the distribution of $x^*/\xi$ is a Gaussian of unit variance†. We can also compute the probability distribution of the ground-state energy $\mathcal{V}^*$ as

$$P(\mathcal{V}^*) = \int \mathrm{d}x \int \mathrm{d}E \; \delta\left( \mathcal{V}^* - \frac{\kappa}{2} x^2 - E \right) P(x, E) \tag{42}$$

where $P(x, E)$ is given in (36). The result is the following: introducing the rescaled energy $u$ as

$$\mathcal{V}^* = E_\mathrm{c} + \frac{1}{2B\delta |E_\mathrm{c}|^{\delta - 1}} \log\left[ \frac{B\delta |E_\mathrm{c}|^{\delta - 1}}{2\pi} \right] + \frac{u}{B\delta |E_\mathrm{c}|^{\delta - 1}} \tag{43}$$

one finds that $u$ is distributed according to the universal Gumbel distribution, equation (7). In particular, the extremely deep states are again exponentially distributed, as $\exp[\mu \mathcal{V}^*/T]$, with $\mu = TB\delta |E_\mathrm{c}|^{\delta - 1}$.

† Note that for small $\kappa$ we have $\kappa \xi^2 \gg |E_\mathrm{c}|$ which justifies our expansion around $E_\mathrm{c}$ in the derivation of $P(\hat{x}^*)$.

### *4.2. Replica approach*

Interestingly, the replica approach with a one-step RSB also leads to the *exact* result. Introducing again the generating function $g(\lambda)$ of $P(E)$, we have

$$\overline{\mathcal{Z}^n} = \sum_{x_1,\dots x_n} \exp\left[ -\frac{\kappa}{2T}\sum_{a=1}^{n} x_a^2 + \sum_x g\left(\frac{1}{T}\sum_{a=1}^{n}\delta_{x,x_a}\right)\right]. \tag{44}$$

The leading saddle point at low temperature is given as in (23) by grouping the $x_a$ in $n/m$ packets of $m$ replicas. Within one packet all the $x_a$ are equal, and the Gaussian integration over the position of each packet gives

$$\overline{\mathcal{Z}^n} = \exp\frac{n}{m}\left[\frac{1}{2}\log\frac{2\pi T}{\kappa m} + g\left(\frac{m}{T}\right)\right] \equiv \exp -\frac{n}{T} f(\rho) \tag{45}$$

with $\rho = m/T$. Looking for the extremum of $f$ as a function of $\rho$ we find, in the limit $\kappa \to 0$ (and with $\delta > 1$),

$$\rho^* = \delta B^{1/\delta}\left(\log\frac{1}{\sqrt{\kappa}}\right)^{(\delta-1)/\delta} \equiv B\delta|E_c|^{\delta-1} \tag{46}$$

where $E_c$ is given by equation (38). It is easy to check that the free energy $f(\rho^*)$ precisely reproduces the above result for the ground-state energy obtained directly, equation (43). As explained above, the calculation of the quantities $Y_k$ within the replica method indicates that the low energy states are exponentially distributed with a parameter given by $\rho^* = m/T$. Hence, comparing (43) and (46), we see that the replica method indeed predicts the correct statistics of deep states. The replica method also allows us to calculate

$$\overline{P(x)} = \sum_{x_2,\dots x_n} \exp\left[-\frac{\kappa}{2T}\sum_{a=1}^{n} x_a^2 + \sum_x g\left(\frac{1}{T}\sum_{a=1}^{n}\delta_{x,x_a}\right)\right]\Bigg|_{x_1=x}. \tag{47}$$

Within the above one-step solution, this immediately leads to the following Gaussian result:

$$\overline{P(x)} = \sqrt{\frac{\kappa\rho^*}{2\pi}}\exp -\frac{\kappa\rho^* x^2}{2} \tag{48}$$

which is identical to equation (40). The replica method also allows us to discuss the nonzero temperature regime, which is much harder to study directly. As shown above, there is a phase transition towards a 'delocalized' phase where $Y_k \equiv 0$ when $\mu = \rho^* T = 1$†. Note that for any small but nonzero temperature and for $\delta > 1$, the system eventually reaches $\mu > 1$ when $\kappa \to 0$. This can be interpreted as follows: as $\kappa \to 0$, the number of accessible states diverges. But since the difference between the ground state and the first excited state decreases as $|E_c|^{1-\delta}$ (when $\delta > 1$), it does become smaller than $T$ for a sufficiently small $\kappa$, beyond which a large number of quasi-degenerate states contribute to the partition function, as in the high-temperature phase. Only for $\delta = 1$ is there a true transition temperature, independent of $\kappa$ (see [16] for a discussion of this point in a different context). For $\delta < 1$, one expects a first-order phase transition (see above).

Finally, let us note that in the case where the confining potential is harmonic (i.e. equal to $\kappa x^2/2$), the Gaussian *variational* replica method developed in [7, 17] also gives the *exact* result for $\rho^*$.

† Note, however, that there is a true phase transition only in the limit $\kappa \to 0$ or $d \to \infty$, i.e. when the number of degrees of freedom is infinite. Otherwise, the transition for $\mu = 1$ is really a crossover.

### *4.3. Physical interpretation of the replica solution*

Using the replica method, we can also compute higher moments of $P(x)$, such as $\overline{P(x)P(y)}$, etc. One can then show that the replica solution is identical to the following probabilistic construction for $P(x)$ for a given sample:

$$P(x) = \frac{1}{\mathcal{Z}} \sum_{\alpha} w_\alpha \delta_{x,x_\alpha} \tag{49}$$

where the $w_\alpha$ are random weights, chosen with a probability distribution given by equation (13), and the $x_\alpha$ are random variables, independent from the $w$'s, and chosen according to a Gaussian of width $\kappa^{-1}$.

## 5. The random energy model and Burgers turbulence

### *5.1. The Cole–Hopf transformation*

It is well known that the solution of Burgers equation with a random initial velocity field can be expressed as a partition sum of the form equation (34). Let us restrict ourselves for simplicity to one dimension, although, again, generalization to higher dimensions is possible. The Burgers equation in the absence of forcing reads

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = \nu \frac{\partial^2 v}{\partial x^2} \tag{50}$$

where $\nu$ is the viscosity. The initial velocity field $v(x, t = 0)$ will be chosen as $v(x, t = 0) = \frac{\partial E(x)}{\partial x}$. Writing $v = -2\nu \frac{\partial \log \mathcal{Z}}{\partial x}$ allows us to transform the Burgers equation into the following linear diffusion equation (Cole–Hopf transform):

$$\frac{\partial \mathcal{Z}}{\partial t} = \nu \frac{\partial^2 \mathcal{Z}}{\partial x^2} \tag{51}$$

with initial condition $\mathcal{Z}(x, t = 0) = \exp[-E(x)/2\nu]$. The solution thus reads

$$\mathcal{Z}(x_0, t) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{\sqrt{4\pi\nu t}} \exp\left(-\frac{1}{2\nu}\left[\frac{(x - x_0)^2}{2t} + E(x)\right]\right) \tag{52}$$

which is, up to a multiplicative factor, identical to the 'spatial' REM defined by (34) with the following identification:

$$T \to 2\nu \qquad \kappa \to \frac{1}{t}. \tag{53}$$

Physically, the disordered problem associated with this REM is that of a point particle interacting with a (random) pinning potential $E(x)$, attached by a spring to point $x_0$, which is a simplified model for an extended elastic object in a random potential. This model was recently considered in the context of solid friction [18].

Although the two problems, spatial REM on one hand and decaying Burgers turbulence on the other hand, are formally identical, they may differ by the type of questions one wants to address. For instance in turbulence one is interested in the correlations of velocities, which involves knowing the variations of the free energy of the REM (52) when $x_0$ varies. The case where $E(x)$ is random with short-range correlations correspond to a short-range correlated velocity field $v(x, t = 0)$ with a 'blue' spectrum (i.e. $|v(k, t = 0)|^2 \propto k^2$, where $k$ is the Fourier variable) and the small viscosity (large Reynolds) limit corresponds to small temperature in the associated disordered problem.

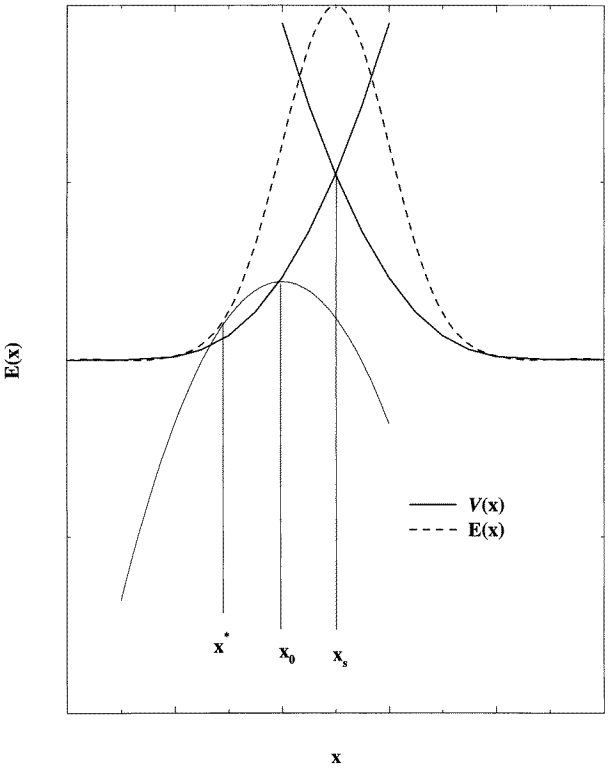


**Figure 2.** Graphical solution of the Burgers equation in the limit of small viscosity, in the neighbourhood of a cusp. The broken curve is the original potential $E(x)$, while the full curve corresponds to the effective potential $\mathcal{V}^*(x)$. The curves actually continue beyond the cusp of $\mathcal{V}^*(x)$, where one metastable and one stable saddle point coexist.

### *5.2. Cusps and shocks*

In the zero-viscosity (or zero-temperature) limit, the partition function (52) can be evaluated by a saddle-point method. For a fixed $x_0$, one looks for the value of $x^*$ such that $\kappa(x_0 - x^*)^2/2 + E(x^*)$ is minimum. The saddle-point construction [8] is graphically explained in figure 2, for a simple profile $E(x)$. For a given $x_0$, one draws as a function of $x$ the parabola $\mathcal{V} - \kappa(x_0 - x)^2/2$ and looks for the minimum value of $\mathcal{V}$, called $\mathcal{V}^*(x_0)$, such that this parabola intersects the curve $E(x)$; calling $x^*$ the intersection point, the saddle-point approximation gives

$$\mathcal{Z}(x_0, t) \simeq \exp\left[-\frac{\mathcal{V}^*(x_0)}{2\nu}\right] \simeq \exp\left(-\frac{1}{2\nu}\left[\frac{\kappa}{2}(x_0 - x^*)^2 + E(x^*)\right]\right). \qquad (54)$$

For large values of $\kappa$, the parabola is very sharp, and there is only one 'optimal' intersection point $x^*$ for each value of $x_0$; to a first approximation, one thus has $\mathcal{Z}(x_0, t) \simeq \exp[-E(x_0)/T]$. On the other hand for very small $\kappa$, which corresponds to the large time limit of the decaying Burgers turbulence, the parabola $\mathcal{V} - \kappa(x_0 - x)^2/2$ is extremely flat and the intersection points will be determined by the extreme (negative) values of the potential $E(x)$. In this limit, the statistics of the effective potential $\mathcal{V}^*$—and thus of the velocity field $v(x, t)$—reflects the statistics of the extreme values of $E(x)$, and is thus, to a large degree, universal. Generically, the solution $x^*$ depends very weakly on $x_0$ and the effective

potential $\mathcal{V}^*(x_0)$ can thus approximately be written as

$$\mathcal{V}^*(x_0) \simeq \frac{\kappa}{2}(x_0 - x^*)^2 + E(x^*) \tag{55}$$

with a *fixed* $x^*$, where $E(x^*)$ corresponds to a particularly 'deep' minimum $x^*$ of the potential $E(x)$. This is the generic situation when one varies $x_0$ locally; it corresponds to a velocity field which is locally linear:

$$v(x_0) = \frac{\mathrm{d}\mathcal{V}^*(x_0)}{\mathrm{d}x_0} = \kappa(x_0 - x^*). \tag{56}$$

(Remember that by definition $\partial\mathcal{V}^*/\partial x^* = 0$.) There exist, however, exceptional values $x_s$ of $x_0$ such that the first intersection of the parabola and the curve $E(x)$ appears simultaneously at two points $x_1^* < x_2^*$: when $x_0$ varies from $x_s - \epsilon$ to $x_s + \epsilon$, the solution $x^*$ jumps from $x_1^*$ to $x_2^*$. This corresponds to a cusp in the minimum value $\mathcal{V}^*$ as a function of $x_0$ (see figure 2). In the language of Burgers turbulence, this is a shock since the velocity $v$ (which is the derivative of $\mathcal{V}^*$) is discontinuous at $x_0 = x_s$.

### *5.3. Decay from an uncorrelated $E(x)$ configuration: Kida's analysis*

Let us now focus on the case where $E(x)$ is randomly distributed with a short-range correlation, and the time $t$ is large, corresponding to a very small $\kappa$. This limit was studied in detail by Kida in the context of Burgers equation [9] (see also [19]). Let us denote by $x_\alpha$ the various values of the intersection points between the parabola and the curve $E(x)$ when one varies $x_0$. After a proper coarse graining we can totally forget about the correlations of $E(x)$, and thus the $\{x_\alpha\}$ are randomly (Poisson) distributed along the $x$-axis. If the distribution of $E$ decays as $\exp -B|E|^\delta$, the extreme-value statistics tells us that the distribution of $E_\alpha \equiv E(x_\alpha)$ is of the Gumbel type. The only delicate point is to understand what is the effective number of independent variables, $M$, appearing in this distribution is. This number depends on $\kappa$ and is determined self-consistently as follows: as $x_0$ departs from $x_\alpha$, at some point (because of the quadratic growing term $\kappa(x_0 - x_\alpha)^2/2$) a better saddle point $x_\beta$ will be preferred. Since the width of the Gumbel distribution is given by

$$\frac{1}{\delta B^{1/\delta}}(\log M)^{\frac{1-\delta}{\delta}} \tag{57}$$

(see equation (6) above), this sets the order of magnitude of the difference between $E_\alpha$ and $E_\beta$, which must also be, by definition, of the order of $\kappa(x_\alpha - x_\beta)^2$. Furthermore, taking the correlation length of the potential $E(x)$ to be one, the effective number of independent variables is given by

$$M = |x_\beta - x_\alpha| \simeq \frac{1}{\sqrt{\kappa\delta B^{1/\delta}}}(\log M)^{\frac{1-\delta}{2\delta}} \tag{58}$$

or, to logarithmic accuracy, and using the correspondance $\kappa \to 1/t$,

$$M \propto \sqrt{t}(\log t)^{\frac{1-\delta}{2\delta}}. \tag{59}$$

Note that by definition, $M$ is also the typical distance beween two shocks $\ell(t)$, which is thus seen to grow as $t^{1/2}$ with logarithmic corrections (these corrections disappear for $\delta = 1$, where the initial potential already possesses the universal exponential tail). This is one of the important results of the original analysis of Kida. Furthermore, since the local slope of the velocity is $\kappa = 1/t$, the maximum velocity is of order

$$v_{\max} = \frac{\ell(t)}{t} \simeq \frac{(\log t)^{\frac{1-\delta}{2\delta}}}{\sqrt{t}} \tag{60}$$

which corresponds to a time dependent Reynolds number

$$Re = \frac{v_{\max}\ell}{\nu} \propto \frac{(\log t)^{(1-\delta)/\delta}}{\nu} \tag{61}$$

which goes to zero (albeit very slowly) when $t \to \infty$ for $\delta > 1$. This is similar to the above remark that for any small temperature, the system goes back into its high-temperature phase when $\kappa \to 0$.

Using this construction, and the full distribution of the $E_\alpha$, Kida was able to obtain directly the large time behaviour of the two-point velocity correlation, $\overline{v(x)v(x+r)}$, which is a universal function once the lengths are expressed in terms of the mean distance between two shocks $\ell(t)$, and the velocities in units of $v_{\max}$ [9]. His result is recalled in the appendix. Let us show how we can obtain precisely the same results using the replica method, which in fact provides the full probability distribution function of $v(x+r) - v(x)$.

### *5.4. The replica analysis*

Let us first note that equation (54) can alternatively (in the limit $\kappa, T \to 0$) be written as an infinite sum:

$$\mathcal{Z}(x_0) = \exp[-\mathcal{V}^*(x_0)/T] = \sum_\alpha w_\alpha \exp\left[-\frac{\kappa(x_0 - x_\alpha)^2}{2T}\right] \tag{62}$$

where $x_\alpha$ are Poisson distributed with an arbitrary (see below) linear density $\sigma$. The $w_\alpha$ are independent random variables again chosen according to the distribution (13), with $\mu$ given by $\mu = T/\Delta(M)$

$$\mu = T\delta B^{1/\delta}(\log M)^{-\frac{1-\delta}{\delta}}. \tag{63}$$

That equation (62) precisely reproduces Kida's construction comes from the fact that, as $T \to 0$, the distribution of weights becomes so broad that the sum determining $\mathcal{Z}(x_0)$ becomes entirely dominated by a single term, which is the one which maximizes $w_\alpha \exp[-\kappa(x_0 - x_\alpha)^2/2]$. Again, the corresponding $x_\alpha$ switches discontinuously as a function of $x_0$, when another value $x_\beta$ suddenly takes over. This construction is independent of the density $\sigma$, provided that $\sigma M \gg 1$ (i.e. in the long time limit).

The crucial point now is that the explicit construction (62) actually gives results which are *identical* to those obtained using a replica representation:

$$\overline{\mathcal{Z}(x_1)\mathcal{Z}(x_2)\ldots\mathcal{Z}(x_n)} = \sum_\pi \exp\left[\tfrac{1}{2}\sum_{a,b=1}^{n} R_{\pi(a),\pi(b)}x_a x_b\right] \tag{64}$$

in the limit $n \to 0$. In the above expression, $\pi$ denotes a permutation of the $n$ replica indices, and the $R_{ab}$ matrix is a one-step RSB matrix [1] with elements $R_{ab} = R_1$ when $a$ and $b$ are in the same diagonal block of size $m$, $R_{ab} = 0$ when $a$ and $b$ are in different blocks, and $R_{aa} = (1-m)R_1$, enforcing the sum rule $\sum_{b=1}^{n} R_{ab} = 0$. It has been shown in [20] (and we recall the main steps of the derivation in the appendix) that the velocity correlation function $\overline{v(x,t)v(y,t)}$ can be computed either directly from equation (62), or using the representation (64).

To test this equivalence of the three constructions of the partition function (54, 62, 64), we show in the appendix that, after a proper choice of length and velocity scales, the $\overline{v(x,t)v(y,t)}$ correlation function is indeed *identical* to the result obtained by Kida (see appendix). The present formalism allows us to extend Kida's results in several directions. For example, the full probability distribution function of $v(x) - v(y)$ can be computed as

in [20]. The problem of decaying Burgers turbulence in higher dimensions can also be addressed.

Let us finally note that the presence of shocks, which manifests itself as a $|x-y|$ singularity in $\overline{v(x,t)v(y,t)}$ at short distances, is intimately connected with the breaking of replica symmetry [20, 5]: for a replica symmetric matrix $R_{aa}=\tilde{R}$, $R_{a\neq b}=R_1$ for all $a\neq b$, $\overline{v(x,t)v(y,t)}$ is *regular* for $x\to y$. As discussed above, these shocks reflect, in the associated disordered problem, the existence of some *metastability* (see figure 2). From a technical point of view, it is interesting to see in this example how metastability is associated with RSB and, as emphasized in [5], with the existence of a short-distance singularity in the effective free-energy $\mathcal{V}^*$. Precisely the same behaviour is obtained via the functional renormalization group (FRG) [21]: a singularity appears in the renormalized correlation function of the effective free energy at scales larger than the 'Larkin length', which is the scale beyond which metastability effects become important. (However, the way to handle the shocks correctly within the FRG is still an open problem [5].)

## 6. Perspectives and other universality classes

As is the case for the central-limit theorem, there are other universality classes, distinct from the Gaussian, when one relaxes the hypothesis of a finite variance or of independent variables (or both) [10]. This is also true for the statistics of extremes, and it is interesting to discuss how this might translate into a replica language. Two main directions can be thought of: independent variables with other types of distributions, or correlated variables.

• Independent random variables. Fréchet and Weibull classes.

Let us first consider the case where the energy levels $E_i$ are still independent, but with a tail for large negative $E$ decaying as a power law, $|E|^{-1-\delta}$. In this case, the extreme values are distributed according to the so-called Fréchet distribution, which is different from the Gumbel distribution (for example, it decays asymptotically as a power law with the same exponent $\delta$). Rescaling $E$ by $M^{1/\delta}$ to keep the gap between the ground state and first excited state finite as $M\to\infty$, we can calculate the quantities $Y_k$ defined in (14). We find, for $M$ large but finite

$$Y_k = 1-\exp-\left(\frac{1}{T\log M}\right)^{\delta} \qquad (\delta<1) \tag{65}$$

*independently* of $k$. (Similar results are obtained for $\delta>1$.) This is clearly different form equation (14). Note that this case cannot be addressed within the replica method without some modifications since all the positive moments $\overline{\mathcal{Z}^n}, n>0$ diverge!

Another universality class corresponds to $E_i$ which are strictly bounded, i.e. $E_i=E_0+\epsilon$, with $\epsilon\geqslant 0$. More precisely, the distribution of $\epsilon$ for $\epsilon\to 0$ is of the form $P(\epsilon)=\epsilon^{\delta}$ for $\epsilon$ small. The resulting distribution of extremes is then called the Weibull distribution. Rescaling the energies by a factor $M^{\delta+1}$, we find through a direct computation that $Y_k$ is nontrivial for all temperatures, i.e. the model is always in a low-temperature phase. For instance, in the case $\delta=0$ one gets $Y_k(T\to\infty)\sim kT^{1-k}$, and $Y_k\sim 1-CT$ for $T\to 0$ ($C$ is Euler's constant). This is again clearly different from the one-step RSB result (14). One might hope that such a situation will lead to a new type of RSB, but the situation seems more complicated. In the particular case $\delta=0$, one finds

$$\overline{\mathcal{Z}^n}=\sum_{i_1,\ldots i_n}{\prod_i}' \frac{T}{\sum_{a=1}^{n}\delta_{i,i_a}} \tag{66}$$

where the product is only over the sites $i$ such that $\sum_{a=1}^{n} \delta_{i,i_a} > 0$. The entropy of the replica symmetric solution becomes negative below the temperature $T_c = 1/e$. Assuming a one-step RSB saddle point for $T < T_c$ leads to a constant average energy, equal to $1/e$; however, the true ground-state energy can be calculated directly and is equal to one. The problem seems to be that the free energy cannot be calculated by a *saddle-point* method in the replica method: replica fluctuations are always important.

In any case, from the point of view of Burgers turbulence, one should keep in mind that initial conditions for the velocity field which do not belong to the exponential universality class considered by Kida will lead to rather different flow structures at long times, even within the class of $E(x)$ functions with local correlations. The bounded case has actually been studied recently by Newman [22].

Let us now turn to the case where the $E_i$ are Gaussian but long-range correlated; for example the case where $E$ depends on a $d$-dimensional space variable $\boldsymbol{x}$, and such that

$$\overline{\tilde{E}(\boldsymbol{q})\tilde{E}(\boldsymbol{q}')} = \frac{\delta(\boldsymbol{q}+\boldsymbol{q}')}{q^{2-\eta}} \tag{67}$$

leading to

$$\overline{(E(\boldsymbol{x}) - E(\boldsymbol{y}))^2} \propto |x-y|^{\max(0,2-d-\eta)}. \tag{68}$$

The case $\eta = 0$, $d = 1$ corresponds to a random walk for $E(x)$, which has been studied in detail both in the context of Burgers turbulence [8], and also as a partly solvable model of a particle in a random potential [14, 23, 24]. The general $\eta, d$ case has not been solved yet. It has been studied by the Gaussian variational replica formalism of [7], which shows [15] that the case $\eta < 2-d$ (corresponding to a growing correlation function (68)) requires 'continuous' RSB, while the case $\eta > 2-d$ only requires a 'one-step' breaking. Independent variables correspond to $\eta = 2$, i.e. a white spectrum for $E(\boldsymbol{q})$. We conjecture here that the case $\eta > 2-d$ belongs to the same (one-step RSB) universality class as the REM ($\eta = 2$). It is actually not difficult to show directly that the quantities

$$c_n = \frac{\overline{\mathcal{Z}^n} - \overline{\mathcal{Z}}^n}{\overline{\mathcal{Z}}^n} \qquad n = 2, 3, \ldots \tag{69}$$

diverge with the system size below a certain $n$-dependent critical temperature which is independent of $\eta$ for $\eta > 2-d$, and identical to those found in the REM. This suggests that the one-step solution indeed remains *exact* for all $\eta > 2-d$. Preliminary numerical simulations [25] seem to confirm this. This points towards a rather natural result, namely the fact that *weak enough correlations* (measured here by $2-\eta$) between the random variables *do not change the universality class for the extreme-value statistics*. This conjecture is supported by a theorem for the case $d = 1$: for all $\eta > 1$, the extreme-value statistics is indeed of the Gumbel type†, while some corrections appear in the marginal case $\eta = 1$. This conjecture is stronger, since it requires that not only the ground state, but also the low-lying energies are independent random variables with an exponential distribution. A special case of this conjecture, supported by numerical simulations, was proposed recently for the $d = 2$ problem with $\eta = 0$, which corresponds to the localization of electrons in a random magnetic field [13]. In this two-dimensional case, the choice $\eta = 2-d = 0$ corresponds to a marginal logarithmic growth of the correlations.

Returning to the one-dimensional case, the situation changes drastically when $\eta < 1$, which corresponds to a typically 'growing' profile $E(x)$. The ratios $c_n$ defined by (69) diverge with the system size for all temperatures, suggesting indeed a change of universality

† See theorem 3.8.2, [3].

class. The only known possibility at present is then to describe the system within a 'continuous' RSB, which can be interpreted as a recursive tree-like construction of the low-lying energy state. In particular, the correlation of the low-lying states have a well known ultrametric structure. How well this ultrametric structure (known to be exact for the case where the dimension of $\boldsymbol{x}$ is infinite) reproduces the distribution and correlations of the low-lying states in finite dimensions is an open problem† which we leave for further studies [25].

It would also be interesting to think of the spin-glass problem from the point of view of the classification of very low-energy states and excitations, which could perhaps provide a natural link between replicas and 'droplet-like' descriptions [26].

## Acknowledgments

We thank A Baldassarri, R Cont and Vik Dotsenko for many interesting discussions. MM thanks the SPhT in the CEA Saclay for its hospitality.

## Appendix

In this appendix we explain briefly how the replica method and the direct probabilistic analysis lead to the same result for the two-point correlation function in decaying Burgers turbulence at large times. We are interested in the case where $E(x)$ has local correlations (see section 5.3), in which case the result of Kida reads

$$\overline{v(x)v(x+r)} = v_{\max}^2 H\left(\frac{r}{\ell(t)}\right) \tag{70}$$

where $\ell(t)$ is the mean distance between two shocks, $v_{\max} = \ell(t)/t$, and

$$H(x) \equiv \frac{1}{\sqrt{2\pi}} \frac{\mathrm{d}}{\mathrm{d}x} x \int_0^\infty \frac{\mathrm{d}y}{\phi(x+y) + \phi(x-y)} \tag{71}$$

and $\phi$ is an error function: $\phi(x) = \int_0^\infty \mathrm{d}z\ \exp(-z^2 + \sqrt{\pi/2}xz)$.

We shall sketch how these results can be obtained from the replica representation (64). The computations are lengthy and already contained in some previous papers. Here we just want to help the interested reader to find his way in the literature in order to obtain the result. One starts from the replicated partition function (64)

$$\overline{\mathcal{Z}(x_1)\mathcal{Z}(x_2)\ldots\mathcal{Z}(x_n)} = \sum_\pi \exp\left[\tfrac{1}{2}\sum_{a,b=1}^n R_{\pi(a),\pi(b)} x_a x_b\right] \tag{72}$$

where the $R_{ab}$ matrix is a one-step RSB matrix [1] with elements $R_{ab} = R_1$ when $a$ and $b$ are in the same diagonal block of size $m$, $R_{ab} = 0$ when $a$ and $b$ are in different blocks, and $R_{aa} = (1-m)R_1$, enforcing the sum rule $\sum_{b=1}^n R_{ab} = 0$. The first step, derived in [20, appendix D], deduces from (72) the correlation between the powers $n/2$ of the partition function

$$\overline{Z(x,t)^{n/2} Z(y,t)^{n/2}} = \frac{2^n}{B(-n/2,-n/2)} \int_0^\infty \frac{\mathrm{d}\mu}{\mu} \mu^{-n/2} \left[\frac{m\sqrt{R_1}}{2\pi} \int \mathrm{d}z\ (e_x + \mu e_y)^m\right]^{n/m}. \tag{73}$$

† Note that the average ground-state energy predicted by the Gaussian *variational* replica theory does not lead back to the exact result [24] in the soluble random walk case $\eta = 0$.

where we have defined

$$e_x \equiv \mathrm{e}^{-mR_1(z-x)^2/2} \qquad e_y = \mathrm{e}^{-mR_1(z-y)^2/2}. \tag{74}$$

Using the general link $v = -2\nu\frac{\partial \log \mathcal{Z}}{\partial x}$, one obtains

$$\overline{v(x)v(y)} = \lim_{n\to 0} \frac{16\nu^2}{n^2}\frac{\partial^2}{\partial x \partial y}\overline{Z(x,t)^{n/2}Z(y,t)^{n/2}} = (2\nu m R_1)^2(g_{11}+g_{12}) \tag{75}$$

where we have defined (the notations are those of [20, appendix B])

$$g_{11} = (1-m)\int_0^\infty \mathrm{d}\mu \frac{\int \mathrm{d}z\,(\mathrm{e}_x + \mu e_y)^{m-2}(x-z)e_x(y-z)e_y}{\int \mathrm{d}z\,(e_x+\mu e_y)^m} \tag{76}$$

and

$$g_{12} = m\int_0^\infty \mathrm{d}\mu\,\frac{(\int \mathrm{d}z\,(e_x+\mu e_y)^{m-1}(x-z)e_x)(\int \mathrm{d}z\,(e_x+\mu e_y)^{m-1}(y-z)e_y)}{\int \mathrm{d}z\,(e_x+\mu e_y)^m}. \tag{77}$$

This expression could also be derived directly without replicas from the infinite sum (62), with the identification $m = \mu$, $R_1 = \kappa/(T\mu)$ as can be seen from [20, formulae (B10), (B11) (where the number $1/(mR_1)$ was called $\delta$). The whole problem is now to evaluate this expression in the limit of large Reynolds, which means small $\mu$ or low $T$. In this regime, using the fact that $m$ scales linearly with $T$ and $R_1$ scales as $1/T^2$, it has been shown in [20, appendix B], that expression (75) reduces to

$$\overline{v(x)v(y)} = \frac{\kappa T}{\mu}\left(\sqrt{\frac{2}{\pi}}\int_0^\infty \mathrm{d}h\,\frac{\mathrm{e}^{-h^2/2}[h^2-d^2/4]}{\mathrm{e}^{d^2/8}[\mathrm{e}^{-hd/2}\mathcal{M}_0(h-\frac{d}{2})+\mathrm{e}^{hd/2}\mathcal{M}_0(-h-\frac{d}{2})]}\right.$$
$$\left. -\frac{d}{\pi}\int_0^\infty \mathrm{d}h\,\frac{\mathrm{e}^{-h^2}}{\mathrm{e}^{d^2/4}[\mathrm{e}^{-hd/2}\mathcal{M}_0(h-\frac{d}{2})+\mathrm{e}^{hd/2}\mathcal{M}_0(-h-\frac{d}{2})]^2}\right) \tag{78}$$

where

$$\mathcal{M}_0(x) = \int_x^\infty \frac{\mathrm{d}z}{\sqrt{2\pi}}\mathrm{e}^{-z^2/2} \qquad d = \frac{|x-y|}{\sqrt{T/(\mu\kappa)}}. \tag{79}$$

So the natural length scale appearing in this solution is $\ell = \sqrt{T/(\mu\kappa)}$. Using (63) and (59), one sees easily that it precisely scales at large times as the average distance between shocks of Kida's analysis. In terms of reduced lengths, we can check that the two distributions (78) and (71) are actually identical.

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